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Historical Perspective

The Principle of Reciprocity

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ABSTRACT

The circumstances surrounding the realisation that NMR signal reception could be quantified in a simple fundamental manner using Lorentz's Principle of Reciprocity are described. The poor signal-to-noise ratio of the first European superconducting magnet is identified as a major motivating factor, together with the author's need to understand phenomena at a basic level. A summary is then given of the thought processes leading to the very simple pseudo-static formula that has been the basis of signal-to-noise calculations for over a generation.

Interview with the author(s).

A video interview with the author(s) associated with this Historical Perspective and the original article can be found in the online version, at doi:10.1016/j.jmr.2011.08.005.

As an undergraduate at Oxford, I studied physics at St. Catherine's College and one of my tutors was Dr. Howard Hill. He worked in Prof. Rex Richards' NMR group, and before leaving for the sunnier climes of California and Varian Associates, he recommended me to Rex as someone who was proficient in electronics – a teenage hobby. Accordingly, in 1968, armed with a freshly minted degree, I started work as a graduate student in the Physical Chemistry Department.

The NMR group had recently taken delivery of Europe's first superconducting NMR magnet (if I remember correctly, 5 T) from the fledgling Oxford Instruments Company. However, plans were soon afoot for a move into the Biochemistry Department with a higher field, "wide bore" (10 mm sample tubes) 7.5 T instrument (Fig. 1), and it soon became clear that I would play a considerable part in designing and building the spectrometer. To prepare, I was given the "bible of NMR" [1] – the book by Anatole Abragam (he obtained his doctorate in Oxford) – and also a paper by Hill and Richards [2] that discussed the various factors affecting signal-to-noise ratio (S/N). This topic was of some concern because the S/N performance of the 5 T magnet was disappointing and no one really knew why.

Now I have always had a need to understand science "from the bottom up" and I found it difficult to relate to both the book and the paper, for they talked about S/N in terms of Q-factor and filling factor, which are not fundamental entities. Further, after some digging I found that both writings assumed solenoidal receiving coils, whereas with a superconducting magnet we were using, perforce, saddle coils. What I *did* relate to, however, was the statement that a free induction decay voltage was induced in the receiving coil by Faraday induction – there I was on home territory.

I reasoned that determining the signal strength should be a simple undergraduate exercise: calculate the electromotive force (EMF) induced in a loop of electrical conductor by a small rotating nuclear magnet **m**. After all, **m** produces a magnetic field and the EMF is merely the rate of change of flux linkage through the receiving coil. Following my standard undergraduate electricity and magnetism text "Bleaney and Bleaney" [3] (an Oxford husband and wife team), I initially approached the problem by dividing the receiving loop into a mesh of small elementary areas – the classic "fishnet" method shown in Fig. 2a. Eq. (5.4) of [3] states that the magnetostatic potential ϕ at a point a distance **r** from a magnetic moment **m**, is

$$\phi = \frac{\mathbf{m}\cdot\mathbf{r}}{4\pi r^3} \tag{1}$$

As magnetic field is the gradient of potential, the flux linkage N through the coil was then the sum of the fluxes passing through each elementary area, as shown in Fig. 2a, or

$$N = -\mu_0 \int_{\text{loop}} \left[\nabla\left(\frac{\mathbf{m}\cdot\mathbf{r}}{4\pi r^3}\right)\right]\cdot d\mathbf{S} \tag{2}$$

where $d\mathbf{S}$ is the vector normal to the surface of an elementary area dS and μ_0 is the permeability of free space. Here, **r** is the distance from the magnet at point P to the elementary area.

At first, I could see no way of simplifying this equation, but then an intuitive relationship occurred to me. The further the magnet was from the loop, the smaller the flux linkage would be, but equally, *if* a current I were made temporarily to flow in the coil, thereby generating a field $\mathbf{B}_1$, the further the magnet was from

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Fig. 1. D. Hoult in Oxford's Biochemistry Department – sample loading ca. 1975.

the coil the smaller the $\mathbf{B}_1$ field would be at the magnet's location P. Clearly, there was a symmetry here and I already had the mathematics at hand to explore it! A current flowing round a loop can be considered to be the sum of currents flowing round a mesh of elementary loops, as shown in Fig. 2b. Knowing that the magnetic moment of current I flowing round an elementary area $d\mathbf{S}$ is $Id\mathbf{S}$, integrating over the mesh I could then use the same mathematical construct and write

$$\mathbf{B}_1 = \mu_0 I \int_{\text{loop}} \nabla \left[\frac{d\mathbf{S} \cdot \mathbf{r}}{4\pi r^3} \right] \quad (3)$$

The dependences on distance $\mathbf{r}$ are the same in Eqs. (2) and (3), apart from the minus sign, which is missing in Eq. (3) because the direction of $\mathbf{r}$ has been reversed. I felt I was on to something.

If I was correct and $N = CmB_1$ (C is a constant), it was clear that the scalar product $\mathbf{B}_1 \cdot \mathbf{m}$ would be needed, if only because N is a scalar and $\mathbf{B}_1$ is a vector. To compare $\mathbf{B}_1 \cdot \mathbf{m}$ with Eq. (2) I did it the hard way, laboriously expanding the two equations in Cartesian coordinates by hand, each equation generating 12 terms. (It takes a few seconds nowadays using a symbolic mathematics programme.) However, once I was satisfied I had made no mistakes, the result confirmed my intuition and I obtained

$$N = -\frac{\mathbf{B}_1}{I} \cdot \mathbf{m} \quad (4)$$

Finally, allowing for an extensive NMR sample and the fact that the induced EMF is proportional to the rate of change of flux linkage, I obtained the equation in "Hoult and Richards" for the NMR free induction decay amplitude ξ, namely

$$\xi = -\int_{\text{sample}} \frac{\partial}{\partial t} (\widehat{\mathbf{B}}_1 \cdot \mathbf{M}_0) dV_s \quad (5)$$

where $\widehat{\mathbf{B}}_1$ is the hypothetical field at a magnetic moment due to unit loop current, M_0 is the nuclear magnetisation and V_s is the sample volume.

As far as the noise was concerned, that was standard undergraduate fare. I had been taught about noise in great detail by another tutor at St. Catherine's, Neville Robinson, who had written a short monograph on the subject [4]. (Among Neville's many accomplishments were the NMR marginal oscillator and the *Encyclopaedia Britannica's* article on electromagnetism.) It was clearly given by the Nyquist equation

$$N = \sqrt{4kTr_c\Delta f} \quad (6)$$

where k is Boltzmann's constant, T is the coil temperature, r_c its resistance and Δf is the bandwidth of the measuring equipment.

All the above was part of my doctoral thesis written in 1972, but the utility of the derivation was not evident to me until I had begun to meet with senior members of the NMR community, in particular Irving Lowe, Joe Dadok and Paul Lauterbur, and to discuss my work with them. It soon became clear that this "Principle of Reciprocity" method should really be published in JMR. It immediately revealed why the performance of the first superconducting magnets was disappointing, for $\widehat{\mathbf{B}}_1/\sqrt{r_c}$ of a saddle coil was about one third that of a solenoid of comparable volume. It also made transparently clear that, at least for spectroscopy, reduction of the coil temperature and resistance could result in increased signal-to-noise ratio, an insight later used to great effect.

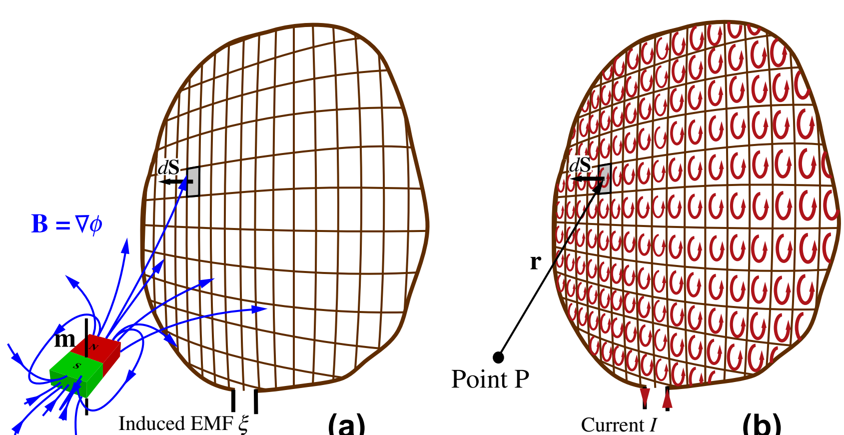

Fig. 2. The classic technique of imposing a mesh on a closed surface whose edge is an electrical conductor. In any integral calculation, the area dS of a mesh element tends to zero. Note that the element is often denoted by a vector $d\mathbf{S}$ perpendicular to its plane. In (a), the mesh concept aids calculation of the flux linkage from a nuclear magnetic moment $\mathbf{m}$ at a point P, while in (b), a current I flowing round the surface edge can be considered to comprise a mesh of circulating currents. Each element has a magnetic moment $Id\mathbf{S}$ that creates a magnetic field – in particular at point P.

It is perhaps a measure of my trust in my undergraduate physics education that it never occurred to me that experimental verification might be needed. The derivation was so elementary and the result I had stumbled on so elegant that it had to be right, a seductive but potentially dangerous mind-set. Only later, as imaging promulgated the idea (due to Purcell and Dicke) that the emission of radio waves was the origin of the MR signal, did I feel the need to roll up my sleeves. The experiment was eventually published in JMR in 2001 [5]. It confirmed the accuracy of the reciprocity approach and that the NMR signal is overwhelmingly due to Faraday induction rather than coherence-brightened spontaneous emission.

In 1972, the idea that NMR could be anything other than a near-field method was unthinkable. One must always look at the underlying conditions of any theory and the derivation reproduced above is firmly and, appropriately for its time, pseudo-static. More recently, however, it has been inappropriately used in imaging when dimensions approach a wavelength. Reciprocity, as formulated (I learnt much later) by Lorentz must be used in this situation, and as most people know by now, Lorentz's construct is strictly for the laboratory frame. Consequently, for ultra-high field MR imaging use, an extension to the *negatively* rotating frame is needed.

In conclusion, I was "hooked" on imaging from the first papers in 1974 onwards, and as field strengths were then a fraction of a tesla, I was soon playing with solenoids, tuned to 4 MHz, wrapped around my head. Of course, this caused great merriment, but when I saw the Q-factor drop from 2000 to 400, I knew the coil resistance had increased and that my electrically conducting head was the cause. (I had moved to a biochemistry department and was expected to know such things.) I showed my results to Paul Lauterbur on one of his summer visits and he immediately said "Let's collaborate." So was born a long relationship and another well-known JMR paper that sprang directly from the work in "Hoult and Richards", but that is another story.

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